# Upper bounds for multiplicities in a tensor product of representations and in a restriction of group representations to a subgroup



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# Upper bounds for multiplicities in a tensor product of representations and in a restriction of group representations to a subgroup

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**Abstract.** Upper bounds for multiplicities of irreducible representations of a semisimple compact Lie group in the tensor product of representations and multiplicities of irreducible representations of a subgroup in an irreducible representation of a group are derived. Pairs of groups and subgroups appear from the representation theory of non-compact groups.

## 1. Introduction

The tensor products of group representations and a restriction of representations of a group into its subgroup are of great importance for contemporary theoretical physics. In this paper we deal with multiplicities of irreducible representations in the decomposition of tensor products and of restrictions. The importance of the problem of multiplicities in physics is seen for example from the fact that physicists derive tables for them (McKay and Patera 1981, Wybourne 1970). However, tables can contain only the representations $\omega$ for which $\dim \omega < N$, $N$ is a fixed integer.

Here we are concerned with another aspect of the multiplicity problem. Namely, we give upper bounds for multiplicities in the tensor products and in a restriction. Some results of this type are known (Boerner 1955, Godement 1952, Harish-Chandra 1954, Wybourne 1970). We derive new results which concern, mostly, degenerate representations (with zero components in highest weights).

Our method uses the relationship between finite dimensional and induced representations of the semisimple non-compact Lie group G. The reciprocity theorem permits us to estimate a multiplicity of an irreducible representation of the maximal compact subgroup K in an induced representation of G. The theorem on the finite dimensional sub-representation (cf § 2 below) defines the induced representation which contains the fixed finite dimensional representation of G. A combination of these theorems leads to the estimates for multiplicities of irreducible representations of K in finite dimensional representations of G. The complexification $G_c$ of G has the compact real form $G_k$. The groups G and $G_k$ have the same finite dimensional representations. Therefore, for finite dimensional representations the reduction $G \to K$ is equivalent to the reduction $G_k \to K$. If G is a complex semisimple Lie group then $G_k = K \otimes K$ and we have the reduction $K \otimes K \to K$ (diagonal imbedding). This reduction leads to the decomposition of the tensor product of representations.

The triples $(G, G_k, K)$ for classical Lie groups are given in table 1. The pairs $G_k \supset K$ for special Lie groups are enumerated in table 2.

**Table 1.** The triples $(\mathrm{G}, \mathrm{G}_k, \mathrm{K})$.

| G | $\mathrm{G}_k$ | K | M |
|---|---|---|---|
| $\mathrm{SL}(n, C)$ | $\mathrm{SU}(n)\times\mathrm{SU}(n)$ | $\mathrm{SU}(n)$ | $\mathrm{U}(1)\times\ldots\times\mathrm{U}(1)$ |
| $\mathrm{SO}(n, C)$ | $\mathrm{SO}(n)\times\mathrm{SO}(n)$ | $\mathrm{SO}(n)$ | $\mathrm{SO}(2)\times\ldots\times\mathrm{SO}(2)$ |
| $\mathrm{Sp}(n, C)$ | $\mathrm{Sp}(n)\times\mathrm{Sp}(n)$ | $\mathrm{Sp}(n)$ | $\mathrm{SO}(2)\times\ldots\times\mathrm{SO}(2)$ |
| $\mathrm{SL}(n, R)$ | $\mathrm{SU}(n)$ | $\mathrm{SO}(n)$ | $\mathrm{Z}$ |
| $\mathrm{SU}^*(2n)$ | $\mathrm{SU}(2n)$ | $\mathrm{Sp}(n)$ | $\mathrm{Sp}(1)\times\ldots\times\mathrm{Sp}(1)$ |
| $\mathrm{SU}(p, q)$ | $\mathrm{SU}(p+q)$ | $\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$ | $\mathrm{S}(\mathrm{U}(p-q)\times\mathrm{U}(1)\times\ldots\times\mathrm{U}(1))$ |
| $\mathrm{U}(p, q)$ | $\mathrm{U}(p+q)$ | $\mathrm{U}(p)\times\mathrm{U}(q)$ | $\mathrm{U}(p-q)\times\mathrm{U}(1)\times\ldots\mathrm{U}(1)$ |
| $\mathrm{SO}_0(p, q)$ | $\mathrm{SO}(p+q)$ | $\mathrm{SO}(p)\times\mathrm{SO}(q)$ | $\mathrm{SO}(p-q)\times\mathrm{Z}$ |
| $\mathrm{SO}^*(2n)$ | $\mathrm{SO}(2n)$ | $\mathrm{U}(n)$ | $\mathrm{SU}(2)\times\ldots\times\mathrm{SU}(2)$, $n=2k$<br>$\mathrm{SU}(2)\times\ldots\times\mathrm{SU}(2)\times\mathrm{U}(1)$, $n=2k+1$ |
| $\mathrm{Sp}(n, R)$ | $\mathrm{Sp}(n)$ | $\mathrm{U}(n)$ | $\mathrm{Z}$ |
| $\mathrm{Sp}(p, q)$ | $\mathrm{Sp}(p+q)$ | $\mathrm{Sp}(p)\times\mathrm{Sp}(q)$ | $\mathrm{Sp}(p-q)\times\mathrm{Sp}(1)\times\ldots\times\mathrm{Sp}(1)$ |

**Table 2.** The pairs $(\mathrm{G}_k, \mathrm{K})$ for special Lie groups.

| $\mathrm{G}_k$ | $\mathrm{E}_6$ | $\mathrm{E}_6$ | $\mathrm{E}_6$ | $\mathrm{E}_6$ | $\mathrm{E}_7$ | $\mathrm{E}_7$ |
|---|---|---|---|---|---|---|
| K | $\mathrm{Sp}(4)$ | $\mathrm{SU}(6)\times\mathrm{SU}(2)$ | $\mathrm{SO}(10)$ | $\mathrm{F}_4$ | $\mathrm{SU}(8)$ | $\mathrm{SO}(12)\times\mathrm{SU}(2)$ |

| $\mathrm{G}_k$ | $\mathrm{E}_7$ | $\mathrm{E}_8$ | $\mathrm{E}_8$ | $\mathrm{F}_4$ | $\mathrm{F}_4$ | $\mathrm{G}_2$ |
|---|---|---|---|---|---|---|
| K | $\mathrm{E}_6$ | $\mathrm{SO}(16)$ | $\mathrm{E}_7\times\mathrm{SU}(2)$ | $\mathrm{Sp}(3)\times\mathrm{SU}(2)$ | $\mathrm{SO}(9)$ | $\mathrm{SU}(2)\times\mathrm{SU}(2)$ |

## 2. Relationship between induced and finite dimensional representations of a non-compact semisimple Lie group

Let G be a non-compact semisimple connected Lie group, and K its maximal compact subgroup. Let $\mathrm{G}=\mathrm{ANK}$ be an Iwasawa decomposition of G, where A is a commutative subgroup, and N is a nilpotent subgroup (Warner 1972). If M is a centraliser of A in K then $\mathrm{P}=\mathrm{ANM}$ is a minimal parabolic subgroup of G. Subgroups M for classical simple Lie groups are given in table 1. A subgroup $\mathrm{P}'$ of G, which contains P, is called parabolic. The decomposition $\mathrm{P}=\mathrm{ANM}$ of P defines the decomposition $\mathrm{P}'=\mathrm{ANM}'$, $\mathrm{M}\subset\mathrm{M}'\subset\mathrm{K}$, of $\mathrm{P}'$ (Warner 1972, Klimyk 1979). Subgroups $\mathrm{M}'$ which will be used below are given in table 3. We give some explanation for table 3. The subgroup $\mathrm{M}_i$ of $\mathrm{Sp}(n, C)$ consists of the matrices

$$\mathrm{diag}(g, u_1, u_2, \ldots, u_i) \qquad g\in\mathrm{Sp}(n-2i) \qquad u_j\in\mathrm{SO}(2).$$

The subgroups $M_i$ of $\mathrm{GL}(n, C)$ and $\mathrm{SO}(n, C)$ are of the same structure. In the case of $\mathrm{U}(p, q)$ the subgroup $\mathrm{M}_i$ consists of the matrices

$$\mathrm{diag}(g_1, u_1, \ldots, u_i, u_i, \ldots, u_1, g_2)$$
$$g_1\in\mathrm{U}(p-i) \qquad g_2\in\mathrm{U}(q-i) \qquad u_j\in\mathrm{U}(1).$$

We have the same structure of $\mathrm{M}_i$ for $\mathrm{SO}_0(p, q)$ and $\mathrm{Sp}(p, q)$. For $\mathrm{SO}_0(p, q)$, $u_j=\pm1$, and for $\mathrm{Sp}(p, q)$, $u_j \in \mathrm{Sp}(1)$. The letter $Z$ in table 3 denotes a discrete subgroup. The subgroup $\mathrm{M}_i$ of $\mathrm{GL}(n, R)$ consists of the matrices

$$\mathrm{diag}(g, u_1, u_2, \ldots, u_i) \qquad g \in \mathrm{O}(n-i) \qquad u_j = \pm 1.$$

We have the same structure of $\mathrm{M}_i$ for $\mathrm{Sp}(n, R)$. In the cases of $\mathrm{SU}^*(2n)$ and $\mathrm{SO}^*(2n)$ the subgroup $\mathrm{M}_i$ consists of the matrices $\mathrm{diag}(g, u_1, \ldots, u_i)$, where $g \in \mathrm{Sp}(n-i)$, $u_j \in \mathrm{Sp}(1)$ for $\mathrm{SU}^*(2n)$, and $g \in \mathrm{U}(n-2i)$, $u_j \in \mathrm{SU}(2)$ for $\mathrm{SO}^*(2n)$. A discription of the groups G and their Lie algebras can be found in Helgason (1962).

**Table 3.** The subgroups M′ and **M**.

| G | M′ | **M** |
|---|---|---|
| $\mathrm{GL}(n, C)$ | $\mathrm{M}_i = \mathrm{U}(i) \times \mathrm{U}(1) \times \ldots \times \mathrm{U}(1)$, $i = 1, 2, \ldots, n-1$ | $\mathbf{M}_i = \mathrm{GL}(i, C) \times \mathrm{U}(1) \times \ldots \times \mathrm{U}(1)$ |
| $\mathrm{SO}(n, C)$ | $\mathrm{M}_i = \mathrm{SO}(n-2i) \times \mathrm{SO}(2) \times \ldots \mathrm{SO}(2)$, $i = 1, 2, \ldots$ | $\mathbf{M}_i = \mathrm{SO}(n-2i, C) \times \mathrm{SO}(2) \times \ldots \times \mathrm{SO}(2)$ |
| $\mathrm{Sp}(n, C)$ | $\mathrm{M}_i = \mathrm{Sp}(n-i) \times \mathrm{SO}(2) \times \ldots \times \mathrm{SO}(2)$, $i = 1, 2, \ldots$ | $\mathbf{M}_i = \mathrm{Sp}(n-i) \times \mathrm{SO}(2) \times \ldots \times \mathrm{SO}(2)$ |
| $\mathrm{U}(p, q)$ | $\mathrm{M}_i = \mathrm{U}(p-i) \times \mathrm{U}(q-i) \times \mathrm{U}(1) \times \ldots \times \mathrm{U}(1)$, $i = 1, 2, \ldots$ | $\mathbf{M}_i = \mathrm{U}(p-i, q-i) \times \mathrm{U}(1) \times \ldots \times \mathrm{U}(1)$ |
| $\mathrm{SO}_0(p, q)$ | $\mathrm{M}_i = \mathrm{SO}(p-i) \times \mathrm{SO}(q-i) \times \mathrm{Z}$, $i = 1, 2, \ldots, q$ | $\mathbf{M}_i = \mathrm{SO} \times (p-i, q-i) \times \mathrm{Z}$ |
| $\mathrm{Sp}(p, q)$ | $\mathrm{M}_i = \mathrm{Sp}(p-i) \times \mathrm{Sp}(q-i) \times \mathrm{Sp}(1) \times \ldots \times \mathrm{Sp}(1)$ | $\mathbf{M}_i = \mathrm{Sp}(p-i, q-i) \times \mathrm{Sp}(1) \times \ldots \times \mathrm{Sp}(1)$ |
| $\mathrm{GL}(n, R)$ | $\mathrm{M}_i = \mathrm{O}(n-i) \times \mathrm{Z}$, $i = 1, 2, \ldots, n$ | $\mathbf{M}_i = \mathrm{GL}(n-i, R) \times Z$ |
| $\mathrm{Sp}(n, R)$ | $\mathrm{M}_i = \mathrm{U}(n-i) \times \mathrm{Z}$, $i = 1, 2, \ldots, n$ | $\mathbf{M}_i = \mathrm{Sp}(n-i, R) \times Z$ |
| $\mathrm{SU}^*(2n)$ | $\mathrm{M}_i = \mathrm{Sp}(n-i) \times \mathrm{Sp}(1) \times \ldots \times \mathrm{Sp}(1)$, $i = 1, 2, \ldots$ | $\mathbf{M}_i = \mathrm{SU}^*(2n-2i) \times \mathrm{Sp}(1) \times \ldots \times \mathrm{Sp}(1)$ |
| $\mathrm{SO}^*(2n)$ | $\mathrm{M}_i = \mathrm{U}(n-2i) \times \mathrm{SU}(2) \times \ldots \times \mathrm{SU}(2)$, $i = 1, 2, \ldots$ | $\mathbf{M}_i = \mathrm{SO}^*(2n-4i) \times \mathrm{SU}(2) \times \ldots \times \mathrm{SU}(2)$ |

The subgroup $\mathrm{P}' = \mathrm{ANM}'$ with $\mathrm{M}' = \mathrm{M}_i$ will be denoted by $\mathrm{P}_i$. We have the Langlands decomposition $\mathrm{P}_i = \mathrm{A}_i \mathrm{N}_i \mathbf{M}_i$ where $\mathrm{N}_i \subset \mathrm{N}$, $\mathrm{A}_i$ consists of elements $h \in \mathrm{A}$, for which $hm = mh$, $m \in \mathrm{M}_i$, and $\mathbf{M}_i$ is a reductive subgroup which is non-compact if $\mathrm{P}_i \neq \mathrm{P}$. A structural description of $\mathbf{M}_i$ and $\mathrm{N}_i$ can be found in Warner (1972). The subgroups $\mathbf{M}_i$ which will be used below are given in table 3.

Let $\delta$ be a finite dimensional irreducible representation of $\mathbf{M}_i$ in the space H, which coincides with the unit operator $\mathbb{1}$ on a non-compact part of $\mathbf{M}_i$. Let $\gamma$ be a complex character of $\mathrm{A}_i$. If $\mathscr{A}_i$ is a Lie algebra of $\mathrm{A}_i$ (that is $\mathrm{A}_i = \exp \mathscr{A}_i$, $\mathscr{A}_i = \log \mathrm{A}_i$), then $\gamma(h) = \exp(\lambda(\log h))$, $h \in \mathrm{A}_i$, where $\lambda$ is a linear form on $\mathscr{A}_i$. The correspondence

$$hnm \to \gamma(h)\delta(m) \qquad h \in \mathrm{A}_i, \qquad n \in \mathrm{N}_i, \qquad m \in \mathbf{M}_i \tag{1}$$

is a representation of $\mathrm{P}_i$ in H. This representation can be written in another form. The character $\gamma$ of $\mathrm{A}_i$ is extended to that of A equating it to 1 on the complement of $\mathrm{A}_i$. In the same way the linear form $\lambda$ on $\mathscr{A}_i$ can be extended to that on $\mathscr{A} = \log \mathrm{A}$. Since $\mathrm{M}_i = \mathbf{M}_i \cap \mathrm{K}$ the representation (1) can be written as

$$hnm \to \gamma(h)\delta(m) \qquad h \in \mathrm{A} \qquad n \in \mathrm{N} \qquad m \in \mathrm{M}_i.$$

Let $L^2_\delta(\mathrm{K}, \mathrm{H})$ be a Hilbert space of functions $f$ from K to H for which $f(mk) = f(k)$, $m \in \mathrm{M}_i$, with the norm

$$\|f\|^2 = \int_{\mathrm{K}} \|f(k)\|^2_{\mathrm{H}}\, \mathrm{d}k.$$

Our representation of $P_i$ induces the representation $\pi^i_{\delta\lambda}$ of G in $L^2_\delta(\mathrm{K}, \mathrm{H})$. The operators $\pi^i_{\delta\lambda}(g)$, $g \in \mathrm{G}$, are given as

$$\pi^i_{\delta\lambda}(g)f(k) = \delta(h)f(k_g)$$

where $k_g \in \mathrm{K}$ and $h \in \mathrm{A}$ are defined by the Iwasawa decomposition of $kg$, that is $kg = hnk_g$, $n \in \mathrm{N}$ (Warner 1972). If $\mathrm{P}_i = \mathrm{P}$ then the representations $\pi^i_{\delta\lambda}$ constitute the principal non-unitary series. The index $i$ for these representations will be omitted.

The well known reciprocity theorem asserts that a multiplicity of the irreducible unitary representation $\sigma$ of K in the representation $\pi^i_{\delta\lambda}$ of G is equal to a multiplicity of the representation $\delta$ of $\mathrm{M}_i$ in $\sigma$.

Let $\mathscr{G}$ be a Lie algebra of G. The subalgebra $\mathscr{A}$ can be extended to a Cartan subalgebra $\mathscr{H}$ of $\mathscr{G}$. Let $\omega$ be a finite dimensional irreducible representation of G, and $\nu$ a weight of $\omega$ with respect to $\mathscr{H}$. A restriction of $\nu$ onto $\mathscr{A}_i$ is denoted by $\nu_i(\omega)$ and is called a restricted weight. Let $\Lambda_i(\omega)$ be a lowest restricted weight of $\omega$ on $\mathscr{A}_i$ (Klimyk 1979, Lepowsky and Wallach 1973). Let $V$ be a subspace of the carrier space of $\omega$ which is spanned onto the vectors $x$ with the restricted weight $\Lambda_i(\omega)$. It is shown by Lepowsky and Wallach (1973) that $\omega$ realises an irreducible representation of $\mathrm{M}_i$ on $V$. It will be denoted by $\delta$. We say that the representation $\omega$ of G is an extension of the representation $\delta$ of $\mathrm{M}_i$. It is clear that there is an infinite number of non-equivalent extensions of $\delta$. A set of these extensions is denoted by $\Omega^i_\delta$. If $\mathrm{P}_i = \mathrm{P}$ the index $i$ will be omitted. If $\omega \in \Omega^i_\delta$ we denote it as $\omega_\delta$.

*Theorem on finite dimensional subrepresentation.* The representation $\pi^i_{\delta\gamma}$ of G may contain only one finite dimensional subrepresentation of G, this being with a multiplicity not exceeding one. Moreover, $\pi^i_{\delta\lambda}$ may contain as a finite dimensional subrepresentation only the representation $\omega_\delta$ (an extension of $\delta$). The representation $\pi^i_{\delta\lambda}$ contains $\omega_\delta$ as subrepresentation if and only if $\Lambda_i(\omega_\delta) = \lambda$.

For the case $\mathrm{P}_i = \mathrm{P}$ the theorem is proved by Lepowsky and Wallach (1973). For $\mathrm{P}_i = \mathrm{P}$ it is given in Klimyk (1979).

The last sentence of the theorem defines exactly a finite dimensional representation of G which is contained in $\pi^i_{\delta\lambda}$ (if it does). Really, the condition $\Lambda_i(\omega_\delta) = \lambda$ separates in $\Omega^i_\delta$ one representation.

Let us consider a set of the representations $\pi^i_{\delta\lambda}$ of G with fixed $\delta$. Some representations $\pi^i_{\delta\lambda}$ contain finite dimensional subrepresentations of G. A set of these subrepresentations coincides with $\Omega^i_\delta$. According to the reciprocity theorem the irreducible representation $\sigma$ of K has the same multiplicity in all representations $\pi^i_{\delta\lambda}$ with fixed $\delta$ and $i$. Therefore, a multiplicity of $\sigma$ in any irreducible representation of G from $\Omega^i_\delta$ does not exceed a multiplicity of $\delta$ in $\sigma$. The compact group $\mathrm{G}_k$ corresponds to the group G (cf table 1). The groups G and $\mathrm{G}_k$ have the same finite dimensional irreducible representations. Thus, we have the following theorem.

*Reciprocity theorem for finite dimensional representations.* A multiplicity of the irreducible representation $\sigma$ of K in any finite dimensional irreducible representation $\omega$ of $\mathrm{G}_k$ from $\Omega^i_\delta$ does not exceed a multiplicity of the representation $\delta$ of $M_i$ in $\sigma$.

It is known (Godement 1952) that a multiplicity of the irreducible representation $\sigma$ of K in any finite dimensional irreducible representation $\omega$ of G (and therefore of $G_k$) does not exceed dim $\sigma$. The reciprocity theorem for finite dimensional representations gives more precise estimates. However, we have to describe the sets $\Omega^i_\delta$. They depend on the type of the pair $(G_k, K)$.

## 3. Multiplicities in the tensor product

Let G be a complex semisimple (or reductive) connected Lie group, and K be its maximal compact subgroup. We shall consider G as a real Lie group with a double number of parameters. Every finite dimensional irreducible representation of G has a form $g \to D^{\Lambda'}_g \otimes \overline{D^{\Lambda}_g}$, where $g \to D^{\Lambda'}_g$ and $g \to D^{\Lambda}_g$ are complex analytic representations of G with the highest weights $\Lambda'$ and $\Lambda$, and a bar denotes a complex conjugation. Since K is a real form of the complex group G, a complex analytic irreducible representation of G is irreducible for K. Therefore, the representation $g \to D^{\Lambda'}_g \otimes \overline{D^{\Lambda}_g}$ of G under a restriction onto K gives the tensor product of the irreducible representations $k \to D^{\Lambda'}_k$ and $k \to D^{\bar{\Lambda}}_k$ of K. The representation $k \to D^{\bar{\Lambda}}_k$ is contragradient to the representation $k \to D^{\Lambda}_k$. Thus, a multiplicity of the irreducible representation $k \to D^{\nu}_k$ of K in the tensor product of the representations $k \to D^{\Lambda'}_k$ and $k \to D^{\bar{\Lambda}}_k$ is equal to that of the representation $k \to D^{\nu}_k$ of K in $g \to D^{\Lambda'}_g \otimes \overline{D^{\Lambda}_g}$. Now we have to apply the theorems formulated above.

Let $G = GL(n, C)$. We use the subgroups $\mathbf{M}_i$ and $M_i$ from table 3. A one-dimensional representation $\delta$ of $\mathbf{M}_i$ and $M_i$ which coincides with $\mathbb{1}$ on $GL(i, C)$, is given by $n-i$ integers $\Lambda_1, \Lambda_2, \ldots, \Lambda_{n-i}$. The integer $\Lambda_j$ defines the representation $\varphi \to \exp(i\Lambda_j\varphi)$ of U(1). We characterise the representation $\delta$ of $M_i$ by a set of integers $(\Lambda_1, \Lambda_2, \ldots, \Lambda_{n-i}, 0, \ldots, 0)$ where $(0, \ldots, 0)$ is a null highest weight for $GL(i, C)$ and $U(i)$. The subgroup $A_i$ consists of the matrices

$$\operatorname{diag}(t_1, t_2, \ldots, t_{n-i}, t, \ldots, t) \qquad t_j > 0 \qquad t > 0.$$

The theorem on finite dimensional subrepresentations gives that $\Omega^i_\delta$ consists of the representations $g \to D^{\Lambda'}_g \otimes \overline{D^{\Lambda}_g}$ of $GL(n, C)$ with the highest weights

$$\Lambda' = (m'_1, m'_2, \ldots, m'_{n-i}, 0, \ldots, 0) \qquad \Lambda = (m_1, m_2, \ldots, m_{n-i}, 0, \ldots, 0)$$

for which $m'_j - m_j = \Lambda_j$, $j = 1, 2, \ldots, n-i$.

Let us find irreducible representations $\sigma$ of $K = U(n)$ which contain the representation $\delta$ of $M_i$. These representations of $U(n)$ have the highest weights $(m_{1n}, m_{2n}, \ldots, m_{nn})$, which are defined by Gel'fand-Zetlin patterns

$$\begin{bmatrix} m_{1n} & m_{2n} & \ldots & m_{n-1,n} & m_{nn} \\ & m_{1,n-1} & \ldots & m_{n-1,n-1} & \\ & & \ldots & & \end{bmatrix} \tag{2}$$

for which $(m_{1i}, m_{2i}, \ldots, m_{ii}) = (0, 0, \ldots, 0)$, and sum (over $k$) of the integers $m_{k,n-j}$ and $m_{k,n-j-1}$ is equal to $\Lambda_j$ $(j = 0, 1, \ldots, n-i-1)$, and

$$m_{1,i+1} + \ldots + m_{i+1,i+1} = \Lambda_{n-i}.$$

Therefore, the highest weights $(m_{1n}, \ldots, m_{nn})$ satisfy the conditions

$$\sum_{j=1}^{n} m_{jn} = \sum_{k=1}^{n-i} \Lambda_k \qquad m_{n-i+1,n} = m_{n-i+2,n} = \ldots = m_{i-1,n} = 0.$$

For $\Lambda=(m_1, m_2, \ldots, m_{n-i}, 0, \ldots, 0)$ we have $\bar{\Lambda}=(0, \ldots, 0, -m_{n-i}, \ldots, -m_2, -m_1)$. Thus, reciprocity theorem for finite dimensional representations gives the following result.

*Theorem.* The tensor product of the irreducible representations of $\mathrm{U}(n)$ with the highest weights $(m_1, m_2, \ldots, m_s, 0, \ldots, 0)$ and $(0, \ldots, 0, m'_s, \ldots, m'_2, m'_1)$ decomposes into a sum of the irreducible representations with the highest weights $(m_{1n}, m_{2n}, \ldots, m_{nn})$ for which

$$\sum_k m_k + \sum_k m'_k = \sum_k m_{kn} \tag{3}$$

$$m_{s+1,n} = m_{s+2,n} = \ldots = m_{n-s+1,n} = 0.$$

A multiplicity of the representation $\sigma$ with the highest weight $(m_{1n}, \ldots, m_{nn})$ in this tensor product does not exceed a multiplicity $m$ in $\sigma$ of the one-dimensional representation $\delta$ of the subgroup $\mathrm{diag}(\mathrm{U}(1), \ldots, \mathrm{U}(1), \mathrm{U}(n-s))$, which is equal to the unit operator on $\mathrm{U}(n-s)$ and is characterised by $(m_1+m'_1, m_2+m'_2, \ldots, m_s+m'_s, 0, \ldots, 0)$.

The multiplicity $m$ can be evaluated for example by means of the Gel'fand–Zetlin patterns (2). The formula

$$\mathrm{mult}(\mathrm{D}^{\Lambda'} \otimes \mathrm{D}^{\Lambda} : \mathrm{D}^{\nu}) = \mathrm{mult}(\mathrm{D}^{\Lambda'} \otimes \mathrm{D}^{\bar{\nu}} : \mathrm{D}^{\bar{\Lambda}})$$

can be applied to obtain other multiplicity rules from the theorem.

Let us note that the first part of the theorem can be derived from the known Kronecker product rules (a wide bibliography and useful rules can be found in Black *et al* (1983)). In particular, relation (3) is well known.

Now let $\mathrm{G}=\mathrm{SO}(n, C)$. It is known that the irreducible representations of $\mathrm{SO}(2l+1)$ are self-contragradient. This statement is valid for $\mathrm{SO}(2l)$ if $m_{ln}=0$ in the highest weight $(m_{1n}, m_{2n}, \ldots, m_{ln})$. Repeating the reasoning for $\mathrm{GL}(n, C)$ we obtain the following theorem.

*Theorem.* The tensor product of irreducible representations of $\mathrm{SO}(n)$ with the highest weights $\Lambda=(m_1, m_2, \ldots, m_i, 0, \ldots, 0)$ and $\Lambda'=(m'_1, m'_2, \ldots, m'_i, 0, \ldots, 0)$ decomposes into a sum of the representations with the highest weights $(m_{1n}, m_{2n}, \ldots, m_{2i,n}, 0, \ldots, 0)$ for which a parity of $m_{1n}+m_{2n}+\ldots+m_{2i,n}$ coincides with that of $\Sigma_j\,(m_j+m'_j)$ (for $\mathrm{SO}(2l+1)$ an assertion on a parity is valid if $2i<1-1$). A multiplicity of the irreducible representation $\sigma$ with the highest weight $(m_{1n}, m_{2n}, \ldots, m_{2i,n}, 0, \ldots, 0)$ in this tensor product does not exceed a multiplicity in $\sigma$ of the one-dimensional representation $\delta$ of $\mathrm{M}_i$ which is equal to $\mathbb{1}$ on $\mathrm{SO}(n-2i)$ and coincides with $\varphi \to \exp(m_j - m'_j)\mathrm{i}\varphi$, $j=1, 2, \ldots, i$, on the subgroup $\mathrm{SO}(2)$.

The parity condition for $m_{1n}+\ldots+m_{2i,n}$ is derived from reciprocity theorem in the following way. We have the Cartan decomposition $\mathrm{so}(n, C)=\mathrm{so}(n)+\mathscr{P}$ for the Lie algebra $\mathrm{so}(n, C)$ of $\mathrm{SO}(n, C)$ (Helgason 1962). Since $[\mathrm{so}(n), \mathscr{P}] \subset \mathscr{P}$, a representation of $\mathrm{SO}(n)$ is realised in $\mathscr{P}$. It is an irreducible representation with the highest weight $(1, 1, 0, \ldots, 0)$ (cf Klimyk 1982). The tensor product of the irreducible representations of $\mathrm{SO}(n)$, $n=2l$, with the highest weights $(m_{1n}, m_{2n}, \ldots)$ and $(1, 1, 0, \ldots, 0)$ decomposes into a sum of representations with the highest weights $(m'_{1n}, m'_{2n}, \ldots)$ for which the integers $m'_{1n}+m'_{2n}+\ldots$ have the same parity as $m_{1n}+m_{2n}+\ldots$ does. This statement is valid for the group $\mathrm{SO}(2l+1)$ if the two last coordinates of the highest weight

$(m_{1n}, m_{2n}, \ldots)$ are equal to 0. Hence, the results of Klimyk (1983) on infinitesimal operators allow us to conclude that a finite dimensional representation of $SO(n, C)$ which is contained in $\pi^i_{\delta\lambda}$, decomposes into a sum of irreducible representations of $SO(n)$ for which numbers $m_{1n}+m_{2n}+\ldots$ are of the same parity (here we demand that $m_{ln}=0$ if $n=2l+1$). This gives the parity condition of the theorem. Of course, the parity condition can be proved by means of the known Kronecker product rules (without use of reciprocity theorem).

A formulation of the theorem for $Sp(n)$ coincides with that for $SO(n)$ if we replace $SO(2n-2i)$ by $Sp(n-2i)$ (cf table 3).

## 4. Multiplicity in the restrictions $G_k \rightarrow K$

The set $\Omega_\delta$ of irreducible finite dimensional representations $\omega$ of G, which are extensions of the representation $\delta$ of M, are described in table 4. Let us note that we did not take into account the subgroup Z of M in the case of $SO_0(p, q)$. The subgroup Z imposes more strong conditions on $\omega \in \Omega_\delta$. They are complicated and we do not formulate them here. In table 4 the integral parts of $\frac{1}{2}(p+q)$, $\frac{1}{2}(p-q)$ and $\frac{1}{2}n$ are denoted by $l$, $s$ and $k$ respectively. We suppose that $p \geqslant q$. We have to take out $(m)$ and the condition $m_{2k+1}=m$ for $SO^*(2n)$ if $n=2k+1$.

Table 5 describes the sets $\Omega^i_\delta$ of the irreducible representations $\omega$ of G which are extensions of the representation $\delta$ of $M_i$. The subgroups $M_i$ are listed in table 3. Again

**Table 4.** The sets $\Omega_\delta$ for classical Lie groups.

| G | $\omega$ | $\delta$ | Conditions for $\omega \in \Omega_\delta$ |
|---|---|---|---|
| $U(p, q)$ | $(m_1, \ldots, m_{p+q})$ | $(n_1, \ldots, n_{p-q})(\Lambda_1)\ldots(\Lambda_q)$ | $m_i - m_{p+q-i} = \Lambda_i,\quad i=1,2,\ldots,q$<br>$m_{q+j}=n_j,\quad j=1,2,\ldots,p-q$ |
| $SO_0(p, q)$ | $(m_1, \ldots, m_l)$ | $(n_1, \ldots, n_s)$ | $m_{l-s+i}=n_i,\quad i=1,2,\ldots,s$ |
| $Sp(p, q)$ | $(m_1, \ldots, m_{p+q})$ | $(n_1, \ldots, n_{p-q})(\Lambda_1)\ldots(\Lambda_q)$ | $m_{2i-1}m_{2i}=\Lambda_i,\quad i=1,2,\ldots,q$<br>$m_{2q-1}=n_j,\quad j=1,\ldots,p-q$ |
| $SU^*(2n)$ | $(m_1, \ldots, m_{2n})$ | $(\Lambda_1)(\Lambda_2)\ldots(\Lambda_n)$ | $m_{2j-1}-m_{2j}=\Lambda_j$ |
| $SO^*(2n)$ | $(m_1, \ldots, m_n)$ | $(\Lambda_1)(\Lambda_2)\ldots(\Lambda_k)(m)$ | $m_{2j-1}-m_{2j}=\Lambda_j,\quad j=1,2,\ldots,k$<br>$m_{2k+1}=m$ |

**Table 5.** The sets $\Omega^i_\delta$ for classical Lie groups.

| G | $\omega$ | $\delta$ | Conditions on $\omega \in \Omega^i_\delta$ |
|---|---|---|---|
| $U(p, q)$ | $(m_1, \ldots, m_{p+q})$ | $(\dot{0})(\Lambda_1)\ldots(\Lambda_{i-1})$ | $m_j - m_{p+q-j}=\Lambda_j,\quad j=1,2,\ldots,i-1$<br>$m_j=0,\quad j=i,i+1,\ldots,p+q-i+1$ |
| $SO_0(p, q)$ | $(m_1, \ldots, m_l)$ | $(\dot{0})$ | $m_{i+j}=0,\quad j=1,2,\ldots,l$ |
| $Sp(p, q)$ | $(m_1, \ldots, m_{p+q})$ | $(\dot{0})(\Lambda_1)\ldots(\Lambda_i)$ | $m_{2j-1}-m_{2j}=\Lambda_j,\quad j=1,2,\ldots,ip+q$<br>$m_{2i+j}=0,\quad j=1,2,\ldots,p-q$ |
| $SL(n, R)$ | $(m_1, \ldots, m_n)$ | $(\dot{0})$ | $m_j=0,\quad j=i+1,i+2,\ldots,n$ |
| $Sp(n, R)$ | $(m_1, \ldots, m_n)$ | $(\dot{0})$ | $m_j=0,\quad j=i+1,i+2,\ldots,n$ |
| $SU^*(2n)$ | $(m_1, \ldots, m_{2n})$ | $(\dot{0})(\Lambda_1)\ldots(\Lambda_i)$ | $m_{2j-I}-m_{2j}=\Lambda_j,\quad j=1,2,\ldots,i$<br>$m_j=0,\quad j=2i+1,2i+2,\ldots,2n$ |
| $SO^*(2n)$ | $(m_1, \ldots, m_n)$ | $(\Lambda_1)\ldots(\Lambda_i)$ | $m_{2j-I}-m_{2j}=\Lambda_j,\quad j=1,2,\ldots,i$<br>$m_j=0,\quad j=2i+1,2i+2,\ldots,n$ |

we did not take into account the discrete subgroups Z of $M_i$. The highest weights $(0, \ldots, 0)$ are denoted by $(\dot{0})$ in table 5. The one-dimensional representation $\varphi \to \exp(i\Lambda_j\varphi)$ of $U(1) \sim SO(2)$ is denoted by $(\Lambda_j)$. The symbol $(\Lambda_j)$ with non-negative integer $\Lambda_j$ is used also to label the irreducible representations of $Sp(1) \sim SU(2)$.

The set $\Omega_\delta$ for the group $U(p, q)$ is given in Klimyk (1979, statement 6.1). The details of derivation are given there. The results of tables 4 and 5 are obtained in the same manner.

It is easy to see that the sets $\Omega_\delta$ and $\Omega^i_\delta$ are infinite. According to reciprocity theorem for finite dimensional representations a multiplicity of a fixed irreducible representation $\sigma$ of K in any irreducible representation $\omega \in \Omega^i_\delta$ of $G_k$ does not exceed a fixed integer $m = \text{mult}\,(\sigma|_{M_i}\colon \delta)$. This result can be formulated independently of the sets $\Omega^i_\delta$. In the case of the triples $(G_k, K, M)$ we have the following theorem.

*Theorem.* Let $(G_k, K, M)$ be a triple from table 1 for which G is one of the groups $U(p, q), SO_0(p, q), Sp(p, q), SU^*(2n), SO^*(2n)$, and $\omega$ be an irreducible representation of $G_k$ with the highest weight $(m_1, m_2, m_3, \ldots)$. Then for any irreducible representation $\sigma$ of K we have the estimate

$$\text{mult}(\omega|_K\colon \sigma) \leq \text{mult}(\sigma|_M\colon \delta)$$

where $\delta$ is the irreducible representation of M which has the highest weight

$$(m_{q+1}, m_{q+2}, \ldots, m_p)(m_1 + m_{p+q})(m_2 + m_{p+q-1}) \ldots (m_q + m_{p+1}) \tag{4}$$

for the reduction $U(p+q) \downarrow U(p) \times U(q)$,

$$(m_{l-s+1}, m_{l-s+2}, \ldots, m_l) \tag{5}$$

for the reduction $SO(p+q) \downarrow SO(p) \times SO(q)$, where $l$ and $s$ are integral parts of $\frac{1}{2}(p+q)$ and $\frac{1}{2}(p-q)$ respectively,

$$(m_{2q+1}, \ldots, m_{p+q})(m_1 + m_2)(m_3 - m_4) \ldots (m_{2q-1} - m_{2q}) \tag{6}$$

for the reduction $Sp(p+q) \downarrow Sp(p) \times Sp(q)$,

$$(m_1 - m_2)(m_3 - m_4) \ldots (m_{2i-1} - m_{2i}) \tag{7}$$

for the reduction $SU(2n) \downarrow Sp(n)$ and

$$(m_1 - m_2)(m_3 - m_4) \ldots (m_{2k-1} - m_{2k})(m_{2k+1}) \tag{8}$$

for the reduction $SO(2n) \downarrow U(n)$ where $k$ is an integral part of $\frac{1}{2}n$, and $m_{2k+1}$ has to be taken out if $n = 2k$. In formulae (4)–(8) $m_1, m_2, \ldots$ are the coordinates of the highest weight of $\omega$.

Some coordinates of the highest weights of the representations $\omega$ of $G_k$ which belong to $\Omega^i_\delta$ are equal to 0 (cf table 5). For the irreducible representations $\sigma$ of K which are contained in $\omega|_K$ we have the estimate

$$\text{mult}(\omega|_K\colon \sigma) \leq \text{mult}(\sigma|_{M^i}\colon \delta) \qquad \omega \in \Omega^i_\delta. \tag{9}$$

According to table 5 some coordinates of the highest weight of $\delta$ are also equal to 0. Then the representations $\sigma$ of K for which $\text{mult}(\sigma|_{M_i}\colon \delta) \neq 0$ have zero coordinates in the highest weights. This and formula (9) mean that zero coordinates of the highest weight of $\omega$ lead to zero coordinates for $\sigma$. The zero coordinates of highest weights of the representations $\sigma$ of K, which are contained in the representation $\omega$, are listed

**Table 6.** The zero coordinates for the representations $\sigma$ of K.

| G | Representations $\omega$ of $G_k$ | Highest weight of $\sigma$ | Zero coordinates for $\sigma$ |
|---|---|---|---|
| $U(p, q)$ | $(m_1, \ldots, m_{p+q})$ $m_i = m_{i+1} = \ldots = m_{p+q-i+1} = 0$ | $(n_1, \ldots, n_p)(n'_1, \ldots, n'_q)$ | $n_i, n_{i+1}, \ldots, n_{p-i+1}, n'_i, \ldots, n'_{q-i+1}$ |
| $SO_0(p, q)$ | $(m_1, \ldots, m_k)$ $m_{i+1} = m_{i+2} = \ldots = m_k = 0$ | $(n_1, \ldots, n_p)(n'_1, \ldots, n'_q)$ | $n_{2i+1}, \ldots, n_p, n'_{2i+1}, \ldots, n'_q$ |
| $Sp(p, q)$ | $(m_1, \ldots, m_{p+q})$ $m_{2i+1} = \ldots = m_{p+q} = 0$ | $(n_1, \ldots, n_p)(n'_1, \ldots, n'_q)$ | $n_{2i+1}, \ldots, n_p, n'_{2i+1}, \ldots, n'_q$ |
| $SL(n, R)$ | $(m_1, \ldots, m_n)$ $m_{i+1} = \ldots = m_n = 0$ | $(n_1, \ldots, n_s)$ | $n_{i+1}, n_{i+2}, \ldots, n_s$ |
| $Sp(n, R)$ | $(m_1, \ldots, m_n)$ $m_{i+1} = \ldots = m_n = 0$ | $(m_{1n}, \ldots, m_{nn})$ | $m_{i+1,n}, m_{i+2,n}, \ldots, m_{n-i,n}$ |
| $SU^*(2n)$ | $(m_1, \ldots, m_{2n})$ $m_{2i+1} = \ldots = m_{2n} = 0$ | $(m_{1n}, \ldots, m_{nn})$ | $m_{2i+1,n}, \ldots, m_{nn}$ |
| $SO^*(2n)$ | $(m_1, \ldots, m_n)$ $m_{2i+1} = \ldots = m_n = 0$ | $(m_{1n}, \ldots, m_{nn})$ | $m_{2i+1,n}, \ldots, m_{n-2i,n}$ |

in table 6. The results of table 6 can be easily proved. Really, it follows from table 3 that the semisimple parts of K and $M_i$ are groups of the same series. Therefore, the representations $\sigma$ of K, for which $\operatorname{mult}(\sigma|_{M_i}: \delta) \neq 0$, can be found by means of Gel'fand-Zetlin patterns. This leads to the results of table 6.

The assertion of table 6 is a weak form of reciprocity theorem for the triple $(G_k, K, M_i)$. Let us formulate other results which can be obtained from this theorem.

*Theorem.* Let $(G_k, K, M_i)$ be a triple defined by table 3, and $\omega$ be a representation of $G_k$ from table 6. Then for any irreducible representation $\sigma$ of K we have the estimate

$$\operatorname{mult}(\omega|_K: \sigma) \leq \operatorname{mult}(\sigma|_{M_i}: \delta) \tag{10}$$

where $\delta$ is the irreducible representation of $M_i$ and its highest weight is defined by $\omega$ and by the pair $(G_k, K)$:

$$\begin{array}{ll}
U(p+q)\downarrow U(p)\times U(q) & (\dot{0})(m_1+m_{p+q})(m_2+m_{p+q-1})\ldots(m_i+m_{p+q-i+1}) \\
SO(p+q)\downarrow SO(p)\times SO(q) & (\dot{0}) \\
Sp(p+q)\downarrow Sp(p)\times Sp(q) & (\dot{0})(m_1-m_2)(m_3-m_4)\ldots(m_{2i-1}-m_{2i}) \\
SU(n)\downarrow SO(n) & (\dot{0}) \\
Sp(n)\downarrow U(n) & (\dot{0}) \\
SU(2n)\downarrow Sp(n) & (\dot{0})(m_1-m_2)(m_3-m_4)\ldots(m_{2i-1}-m_{2i}) \\
SO(2n)\downarrow U(n) & (m_1-m_2)(m_3-m_4)\ldots(m_{2i-1}-m_{2i}).
\end{array}$$

The discrete subgroups Z of $M_i$ were not taken into account in this theorem. Let us note that the theorem admits simpler (but more rough) formulation. Let $M'_i$ be a semisimple part (or $U(p-i)\times U(q-i)$ if $K = U(p)\times U(q)$) of $M_i$. Then the formula

$$\operatorname{mult}(\omega|_K: \sigma) \leq \operatorname{mult}(\sigma|_{M'_i}: \dot{0}) \tag{11}$$

is the weak form of formula (10). Here $\dot{0}$ denotes the unit representation of $M_i'$. Since $M_i'$ and K are groups of the same series (cf table 3), the multiplicity mult $(\sigma|_{M_i'}: \dot{0})$ can be evaluated with a help of Gel'fand-Zetlin patterns (cf Klimyk and Gruber 1979).

## 5. Conclusions

We have obtained estimates for multiplicities of irreducible representations in the tensor product and in the reduction of representations of a group onto its subgroup. To obtain these estimates we have used a part of parabolic subgroups of a semisimple non-compact Lie group. An application of other parabolic subgroups leads to new estimates. Parabolic subgroups of a semisimple non-compact Lie group can be obtained from a root system (cf Warner 1972).

Let us note that some results of this paper can be derived from the known Kronecker product and branching rules.

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